Random matrix theory within superstatistics

A. Y. Abul-Magd
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt
(Received 10 August 2005; published 13 December 2005)

We propose a generalization of the random matrix theory following the basic prescription of the recently suggested concept of superstatistics. Spectral characteristics of systems with mixed regular-chaotic dynamics are expressed as weighted averages of the corresponding quantities in the standard theory assuming that the mean level spacing itself is a stochastic variable. We illustrate the method by calculating the level density, the nearest-neighbor-spacing distributions, and the two-level correlation functions for systems in transition from order to chaos. The calculated spacing distribution fits the resonance statistics of random binary networks obtained in a recent numerical experiment.

DOI: 10.1103/PhysRevE.72.066114 PACS number(s): 05.70.−a, 05.40.−a, 05.45.Mt, 03.65.−w

I. INTRODUCTION

Random matrix theory (RMT) provides a suitable framework to describe quantal systems whose classical counterpart has a chaotic dynamics [1,2]. It models a chaotic system by an ensemble of random Hamiltonian matrices H that belong to one of the three universal classes, namely the Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, and GSE, respectively). The theory is based on two main assumptions: (i) the matrix elements are independent identically distributed random variables, and (ii) their distribution is invariant under unitary transformations. These lead to a Gaussian probability density distribution for the matrix elements, $P(H)\propto\exp[-\eta\,\mathrm{Tr}(H^{\dagger}H)]$. With these assumptions, RMT presents a satisfactory description for numerous chaotic systems. On the other hand, there are elaborate theoretical arguments by Berry and Tabor [3], which are supported by several numerical calculations, that the nearest-neighbor-spacing (NNS) distribution of classically integrable systems should have a Poisson distribution $\exp(-s)$, although exceptions exist.

For most systems, however, the phase space is partitioned into regular and chaotic domains. These systems are known as mixed systems. Attempts to generalize RMT to describe such mixed systems are numerous; for a review please see [4]. Most of these attempts are based on constructing ensembles of random matrices whose elements are independent but not identically distributed. Thus, the resulting expressions are not invariant under base transformation. To the best of our knowledge, the first work in this direction is due to Rosenzweig and Porter [5]. They model the Hamiltonian of the mixed system by a superposition of a diagonal matrix of random elements having the same variance and a matrix drawn from a GOE. Therefore, the variances of the diagonal elements of the total Hamiltonian are different from those of the off-diagonal ones, unlike the GOE Hamiltonian in which the variances of diagonal elements are twice those of the off-diagonal ones. Hussein and Pato [6] used the maximum entropy principle to construct "deformed" random matrix ensembles by imposing different constraints for the diagonal and off-diagonal elements. This approach has been successfully applied to the case of a metal-insulator transition [7]. A recent review of the deformed ensemble is given in [8]. Ensembles of band random matrices, whose entries are equal to zero outside a band of limited width along the principal diagonal, have often been used to model mixed systems [2,9,10]. However, so far in the literature, there is no rigorous statistical description for the transition from integrability to chaos. The field remains open for new proposals.

The past decade has witnessed a considerable interest devoted to the possible generalization of statistical mechanics. Much work in this direction followed Tsallis paper [11]. Tsallis introduced a nonextensive entropy, which depends on a positive parameter q known as the entropic index. The standard Shannon entropy is recovered for $q=1$. Applications of the Tsallis formalism covered a wide class of phenomena; for a review please see, e.g., [12]. Recently, the formalism has been applied to include systems with mixed regular-chaotic dynamics in the framework of RMT [13–18]. This is done by extremizing Tsallis' nonextensive entropy, rather than Shannon's, but again subject to the same constraints of normalization and existence of the expectation value of $\mathrm{Tr}(H^{\dagger}H)$. The latter constraint preserves base invariance. The first attempt in this direction is probably due to Evans and Michael [13]. Toscano *et al.* [14] constructed non-Gaussian ensemble by minimizing Tsallis' entropy and obtained expressions for the level densities and spacing distributions for mixed systems belonging to the orthogonal-symmetry universality class. Bertuola *et al.* [15] expressed the spectral fluctuation in the subextensive regime in terms of the gap function, which measures the probability of an eigenvalue-free segment in the spectrum. A slightly different application of nonextensive statistical mechanics to RMT is due to Nobre and Souza [16]. The nearest-neighbor-spacing distributions obtained in this approach decays as a power law for large spacings. Such anomalous distributions can hardly be used to interpolate between nearly regular systems which have almost exponential NNS distributions and nearly chaotic ones whose distributions behave at large spacing as Gaussians. Moreover, the constraints of normalization and existence of an expectation value for $\mathrm{Tr}(H^{\dagger}H)$ set up an upper limit for the entropic index q beyond which the involved integrals diverge. This restricts the validity of the nonextensive RMT to a limited range near the chaotic phase [17,18].

1539-3755/2005/72(6)/066114(9)/$23.00

Another extension of statistical mechanics is provided by the formalism of superstatistics (statistics of a statistics), recently proposed by Beck and Cohen [19]. Superstatistics arises as weighted averages of ordinary statistics (the Boltzmann factor) due to fluctuations of one or more intensive parameters (e.g., the inverse temperature). It includes Tsallis' nonextensive statistics, for $q \geqslant 1$, as a special case in which the inverse temperature has χ^2 distributions. With other distributions of the intensive parameters, one comes to other more general superstatistics. Generalized entropies, which are analogous to the Tsallis entropy, can be defined for these general superstatistics [20–22]. This formalism has been elaborated and applied successfully to a wide variety of physical problems, e.g., in [23–30].

In a previous paper [31], the concept of superstatistics was applied to model a mixed system within the framework of RMT. The joint matrix element distribution was represented as an average over $\exp[-\eta \,\mathrm{Tr}(H^{\dagger}H)]$ with respect to the parameter η. An expression for the eigenvalue distributions was deduced. Explicit analytical results were obtained for the special case of two-dimensional random matrix ensembles. Different choices of parameter distribution, which had been studied in Beck and Cohen's paper [19] were considered. These distributions essentially led to equivalent results for the level density and NNS distributions. The present paper is essentially an extension of the superstatistical approach of Ref. [31] to random matrix ensembles of arbitrary dimension. The distribution of local mean level densities is estimated by applying the principle of maximum entropy, as done by Sattin [27]. In Sec. II we briefly review the superstatistics concept and introduce the necessary generalization required to express the characteristics of the spectrum of a mixed system into an ensemble of chaotic spectra with different local mean level density. The evolution of the eigenvalue distribution during the stochastic transition induced by increasing the local-density fluctuations is considered in Sec. III. The corresponding NNS distributions are obtained in Sec. IV for systems in which the time-reversal symmetry is conserved or violated. Section V considers the two-level correlation functions. The conclusion of this work is formulated in Sec. VI.

II. FORMALISM

A. Superstatistics and RMT

To start with, we briefly review the superstatistics concept as introduced by Beck and Cohen [19]. Consider a nonequilibrium system with spatiotemporal fluctuations of the inverse temperature β. Locally, i.e., in spatial regions (cells) where β is approximately constant, the system may be described by a canonical ensemble in which the distribution function is given by the Boltzmann factor $e^{-\beta E}$, where E is an effective energy in each cell. In the long-term run, the system is described by an average over the fluctuating β. The system is thus characterized by a convolution of two statistics, and hence the name "superstatistics." One statistics is given by the Boltzmann factor and the other one by the probability distribution $f(\beta)$ of β in the various cells. One obtains Tsallis' statistics when β has a χ^2 distribution, but this is not the only possible choice. Beck and Cohen give several possible examples of functions which are possible candidates for $f(\beta)$. Sattin [27] suggested that, lacking any further information, the most probable realization of $f(\beta)$ will be the one that maximizes the Shannon entropy. Namely, this version of superstatistics formalism will now be applied to RMT.

Gaussian random-matrix ensembles have several common features with the canonical ensembles. In RMT, the square of a matrix element plays the role of energy of a molecule in a gas. When the matrix elements are statistically identical, one expects them to become distributed as the Boltzmann elements. One obtains a Gaussian probability density distribution of the matrix elements

$$P(H) \propto \exp[-\eta \,\mathrm{Tr}(H^{\dagger}H)] \tag{1}$$

by extremizing the Shannon entropy [1,32] subjected to the constraints of normalization and existence of the expectation value of $\mathrm{Tr}(H^{\dagger}H)$. The quantity $\mathrm{Tr}(H^{\dagger}H)$ plays the role of the effective energy of the system, while the role of the inverse temperature β is played by η, being twice the inverse of the matrix-element variance.

Our main assumption is that Beck and Cohen's superstatistics provides a suitable description for systems with mixed regular-chaotic dynamics. We consider the spectrum of a mixed system as made up of many smaller cells that are temporarily in a chaotic phase. Each cell is large enough to obey the statistical requirements of RMT, but has a different distribution parameter η associated with it, according to a probability density $\tilde{f}(\eta)$. Consequently, the superstatistical random-matrix ensemble that describes the mixed system is a mixture of Gaussian ensembles. Its matrix-element joint probability density distributions obtained by integrating distributions of the form in Eq. (1) over all positive values of η with a statistical weight $\tilde{f}(\eta)$ are given by,

$$P(H) = \int_0^{\infty} \tilde{f}(\eta) \frac{\exp[-\eta \,\mathrm{Tr}(H^{\dagger}H)]}{Z(\eta)} d\eta, \tag{2}$$

where $Z(\eta) = \int \exp[-\eta \,\mathrm{Tr}(H^{\dagger}H)] d\eta$. Here we use the "$B$-type superstatistics" [19]. The distribution in Eq. (2) is isotropic in the matrix-element space. Relations analogous to Eq. (1) can also be written for the joint distribution of eigenvalues as well as any other statistic that is obtained from it by integration over some of the eigenvalues, such as the nearest-neighbor-spacing distribution and the level number variance. The distribution $\tilde{f}(\eta)$ has to be normalizable, to have at least a finite first moment

$$\langle \eta \rangle = \int_0^{\infty} \tilde{f}(\eta)\, \eta \, d\eta, \tag{3}$$

and then reduces a delta function as the system becomes fully chaotic.

The random-matrix distribution in Eq. (2) is invariant under base transformation because it depends on the Hamiltonian matrix elements through the base-invariant quantity $\mathrm{Tr}(H^{\dagger}H)$. Factorization into products of individual element distributions is lost here, unlike in the distribution functions of the standard RMT and most of its generalizations for mixed systems. The matrix elements are no more statistically independent. This handicaps one in carrying numerical calculations by the random-number generation of ensembles and forces one to resort to artificial methods as done in [14]. The base invariance makes the proposed random-matrix formalism unsuitable for a description of nearly integrable systems. These systems are often described by an ensemble of diagonal matrices in a presumably fixed basis. For this reason we expect the present superstatistical approach to describe only the final stages of the stochastic transition. The base invariant theory in the proposed form does not address the important problem of symmetry breaking in a chaotic system, where the initial state is modeled by a block diagonal matrix with m blocks, each of which is a GOE [4]. This problem is well described using deformed random-matrix ensembles as in [6] or phenomenologically by considering the corresponding spectra as superpositions of independent subspectra, each represented by a GOE [33]a.

The physics behind the proposed superstatistical generalization of RMT is the following. The eigenstates of a chaotic system are extended and cover the whole domain of classically permitted motion randomly, but uniformly. They overlap substantially, as manifested by level repulsion. There are no preferred eigenstates; the states are statistically equivalent. As a result, the matrix elements of the Hamiltonian in any basis are independently but identically distributed, which leads to the Wigner-Dyson statistics. Coming out of the chaotic phase, the extended eigenstates become less and less homogeneous in space. Different eigenstates become localized in different places and the matrix elements that couple different pairs are no more statistically equal. The matrix elements will no longer have the same variance; one has to allow each of them to have its own variance. But this will dramatically increase the number of parameters of the theory. The proposed superstatistical approach solves this problem by treating all of the matrix elements as having a common variance, not fixed but fluctuating.

B. Eigenvalue distribution

The matrix-element distribution is not directly useful in obtaining numerical results concerning energy-level statistics such as the nearest-neighbor-spacing distribution, the two-point correlation function, the spectral rigidity, and the level-number variance. These quantities are presumably obtainable from the eigenvalue distribution. From (1), it is a simple matter to set up the eigenvalue distribution of a Gaussian ensemble. With $H=U^{-1}EU$, where U is the global unitary group, we introduce the elements of the diagonal matrix of eigenvalues $E=\mathrm{diag}(E_1,\ldots,E_N)$ of the eigenvalues and the independent elements of U as new variables. Then the volume element (4) has the form

$$dH=|\Delta_N(E)|^{\beta}dE\,d\mu(U), \tag{4}$$

where $\Delta_N(E)=\Pi_{n>m}(E_n-E_m)$ is the Vandermonde determinant and $d\mu(U)$ the invariant Haar measure of the unitary group [1,4]. Here $\beta=1$, 2, and 4 for GOE, GUE and GSE, respectively. The probability density $P_\beta(H)$ is invariant under arbitrary rotations in the matrix space. Integrating over U yields the joint probability density of eigenvalues in the form

$$P_\beta(E_1,\ldots,E_N)=\int_0^\infty f(\eta)P_\beta^{(G)}(\eta,E_1,\ldots,E_N)d\eta, \tag{5}$$

where $P_\beta^{(G)}(\eta,E_1,\ldots,E_N)$ is the eigenvalue distribution of the corresponding Gaussian ensemble, which is given by

$$P_\beta^{(G)}(\eta,E_1,\ldots,E_N)=C_\beta|\Delta_N(E)|^\beta\exp\left[-\eta\sum_{i=1}^N E_i^2\right], \tag{6}$$

where C_β is a normalization constant. Similar relations can be obtained for any statistic $\sigma_\beta(E_1,\ldots,E_k)$, with $k<N$, that can be obtained from $P_\beta(E_1,\ldots,E_N)$ by integration over the eigenvalues $E_{k+1},\ldots,E_N$.

In practice, one has a spectrum consisting of a series of levels $\{E_i\}$, and is interested in their fluctuation properties. In order to bypass the effect of the level density variation, one introduces the so-called "unfolded spectrum" $\{\varepsilon_i\}$, where $\varepsilon_i=E_i/D$ and D is the local mean level spacing. Thus, the mean level density of the unfolded spectrum is unity. On the other hand, the energy scale for a Gaussian random matrix ensemble is defined by the parameter η. The mean level spacing may be expressed as

$$D=\frac{c}{\sqrt{\eta}}, \tag{7}$$

where c is a constant depending on the size of the ensemble. Therefore, although the parameter η is the basic parameter of RMT, it is more convenient for practical purposes to consider the local mean spacing D itself instead of η as the fluctuating variable for which superstatistics has to be established.

The new framework of RMT provided by superstatistics should now be clear. The local mean spacing D is no longer a fixed parameter but it is a stochastic variable with probability distribution $f(D)$. Instead, the the observed mean level spacing is just its expectation value. The fluctuation of the local mean spacing is due to the correlation of the matrix elements which disappears for chaotic systems. In the absence of these fluctuations, $f(D)=\delta(D-1)$ and we obtain the standard RMT. Within the superstatistics framework, we can express any statistic $\sigma(E)$ of a mixed system that can, in principle, be obtained from the joint eigenvalue distribution by integration over some of the eigenvalues, in terms of the corresponding statistic $\sigma^{(G)}(E,D)$ for a Gaussian random ensemble. The superstatistical generalization is given by

$$\sigma(E)=\int_0^\infty f(D)\sigma^{(G)}(E,D)dD. \tag{8}$$

The remaining task of superstatistics is the computation of the distribution $f(D)$.

C. Evaluation of the local-mean-spacing distribution

Following Sattin [27], we use the principle of maximum entropy to evaluate the distribution $f(D)$. Lacking detailed information about the mechanism causing the deviation from the prediction of RMT, the most probable realization of $f(D)$

will be the one that extremizes the Shannon entropy

$$S = -\int_0^\infty f(D)\ln f(D)dD \tag{9}$$

with the following constraints:

Constraint 1. The major parameter of RMT is η defined in Eq. (1). Superstatistics was introduced in Eq. (2) by allowing η to fluctuate around a fixed mean value $\langle\eta\rangle$. This implies, in light of Eq. (7), the existence of the mean inverse square of D,

$$\langle D^{-2}\rangle = \int_0^\infty f(D)\frac{1}{D^2}dD. \tag{10}$$

Constraint 2. The fluctuation properties are usually defined for unfolded spectra, which have a unit mean level spacing. We thus require

$$\int_0^\infty f(D)D\,dD = 1. \tag{11}$$

Therefore, the most probable $f(D)$ extremizes the functional

$$F = -\int_0^\infty f(D)\ln f(D)dD - \lambda_1\int_0^\infty f(D)D\,dD - \lambda_2\int_0^\infty f(D)\frac{1}{D^2}dD, \tag{12}$$

where λ_1 and λ_2 are Lagrange multipliers. As a result, we obtain

$$f(D) = C\exp\left[-\alpha\left(\frac{2D}{D_0}+\frac{D_0^2}{D^2}\right)\right], \tag{13}$$

where α and D_0 are parameters, which can be expressed in terms of the Lagrange multipliers λ_1 and λ_2, and C is a normalization constant. We determine D_0 and C by using Eqs. (10) and (11) as

$$D_0 = \alpha\frac{G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)}{G_{03}^{30}\left(\alpha^3|0,1,\frac{3}{2}\right)}, \tag{14}$$

and

$$C = \frac{2\alpha\sqrt{\pi}}{D_0 G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)}. \tag{15}$$

Here $G_{03}^{30}(x|b_1,b_2,b_2)$ is a Meijer's G function defined in the Appendix.

III. LEVEL DENSITY

The density of states can be obtained from the joint eigenvalue distribution directly by integration

$$\rho(E) = N\int\cdots\int P_\beta(E,E_2,\ldots,E_N)dE_2,\ldots dE_N. \tag{16}$$

For a Gaussian ensemble, simple arguments [1,35] lead to Wigner's semicircle law

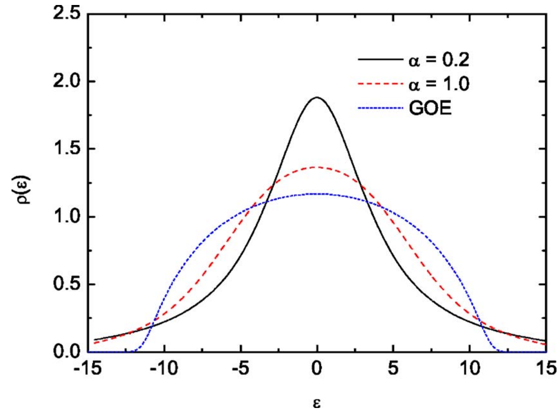

FIG. 1. (Color online) Level density for superstatistical orthogonal ensembles with parameters α=0.2, 1, and ∞ (the GOE limit).

$$\rho_{GE}(E,D) = \begin{cases} \frac{2N}{\pi R_0^2}\sqrt{R_0^2-E^2}, & \text{for } |E| \leq R_0 \\ 0, & \text{for } |E| > R_0 \end{cases}, \tag{17}$$

where D is the mean level spacing, while the prefactor is chosen so that $\rho_{GE}(E)$ satisfies the normalization condition

$$\int_{-\infty}^{\infty}\rho_{GE}(E)dE = N. \tag{18}$$

We determine the parameter R_0 by requiring that the mean level density is $1/D$ so that

$$\frac{1}{N}\int_{-\infty}^{\infty}[\rho_{GE}(E)]^2dE = \frac{1}{D}. \tag{19}$$

This condition yields

$$R_0 = \frac{16N}{3\pi^2}D. \tag{20}$$

Substituting (17) into (8) we obtain the following expression for the level density of the superstatistical ensemble:

$$\rho_{SE}(E,\alpha) = \int_0^{3\pi^2|E|/(16N)} f(D,\alpha)\rho_{GE}(E,D)dD. \tag{21}$$

We could not solve this integral analytically. We evaluated it numerically for different values of α. The results of calculation are shown in Fig. 1. The figure shows that the level density is symmetric with respect to E=0 for all values of α and has a pronounced peak at the origin. However, the behavior of the level density for finite α is quite distinct from the semicircular law. It has a long tail whose shape and decay rate both depend on the choice of the parameter distribution $f(D)$. This behavior is similar to that of the level density of mixed system modeled by a deformed random matrix ensemble [34].

IV. NEAREST-NEIGHBOR-SPACING DISTRIBUTION

The NNS distribution is probably the most popular characteristic used in the analysis of level statistics. In principle, it can be calculated once the joint-eigenvalue distribution is known. The superstatistics generalization of NNS distribution for an ensemble belonging to a given symmetry class is obtained by substituting the NNS distribution of the corresponding Gaussian ensemble $P_{\mathrm{GE}}(s,D)$ for $\sigma^{(G)}(E,D)$ in (7) and integrating over the local mean level spacing D

$$P_{\mathrm{SE}}(s) = \int_0^\infty f(D) P_{\mathrm{GE}}(s,D) dD. \tag{22}$$

Until now, no analytical expression for the NNS distribution could be derived from RMT. What we know is that this distribution is very well approximated by the Wigner surmise [1]. We shall obtain superstatistics for NNS distribution for systems with orthogonal and unitary symmetries by assuming that the corresponding Gaussian ensembles have Wigner distributions for the nearest-neighbor spacings.

Equation (22) yields the following relation between the second moment $\langle D^2\rangle$ of the local-spacing distribution $f(D)$ and the second moment $\langle s^2\rangle$ of the spacing distribution $P_{\mathrm{SE}}(s)$:

$$\langle D^2\rangle = \frac{\langle s^2\rangle}{\langle s^2\rangle_{\mathrm{GE}}}, \tag{23}$$

where $\langle s^2\rangle_{\mathrm{GE}}$ is the mean square spacing for the corresponding Gaussian ensemble. For the distribution in Eq. (13), one obtains

$$\langle D^2\rangle = \frac{G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right) G_{03}^{30}\left(\alpha^3|0,\frac{3}{2},2\right)}{\left[G_{03}^{30}\left(\alpha^3|0,1,\frac{3}{2}\right)\right]^2}. \tag{24}$$

Using the asymptotic behavior of the G function, we find that $\langle D^2\rangle \to 1$ as $\alpha\to\infty$, while $\langle D^2\rangle=2$ (as for the Poisson distribution) when $\alpha=0$. For practical purposes, the expression in Eq. (24) can be approximated with sufficient accuracy by $\langle D^2\rangle \approx 1+1/(1+4.121\alpha)$. Thus, given an experimental or numerical-experimental NNS distibution, one can evaluate the quantity $\langle s^2\rangle$ and estimate the corresponding value of the parameter α by means of the following approximate relation:

$$\alpha \approx 0.243 \frac{\langle s^2\rangle}{\langle s^2\rangle - \langle s^2\rangle_{\mathrm{GE}}}. \tag{25}$$

A. Orthogonal ensembles

Systems with spin-rotation and time-reversal invariance belong to the orthogonal symmetry class of RMT. Chaotic systems of this class are modeled by GOE for which NNS is well approximated by the Wigner surmise

$$P_{\mathrm{GOE}}(s,D) = \frac{\pi}{2D^2} s \exp\left(-\frac{\pi}{4D^2}s^2\right). \tag{26}$$

We now apply superstatistics to derive the corresponding NNS distribution assuming that the local-mean-spacing distribution $f(D)$ is given by Eq. (13). Substituting (26) into (22), we obtain

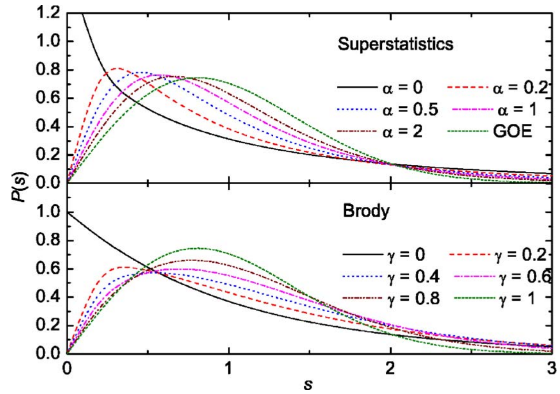

FIG. 2. (Color online) Evolution of NNS distributions obtained by the superstatistics method for systems undergoing a transition from the GOE statistics to the Poissonian, compared with the Brody's distributions.

$$P_{\mathrm{SOE}}(s,\alpha) = \frac{\pi\alpha^2}{2D_0^2 G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)} s G_{03}^{30}\left(\alpha^3 + \frac{\pi\alpha^2}{4D_0^2}s^2 \Big| -\frac{1}{2},0,0\right), \tag{27}$$

where D_0 is given by (14), while the subscript SOE stands for superstatistical orthogonal ensemble.

Because of the difficulties of calculating $G_{0,3}^{3,0}(z|b_1,b_2,b_3)$ at large values of z, we use (say for $z>100$) the large z asymptotic formula given in the Appendix to obtain

$$P_{\mathrm{SOE}}(s,\alpha) \approx \frac{\pi}{2} s \frac{\exp\left[-3\alpha\left(\sqrt[3]{1+\frac{\pi s^2}{4\alpha}}-1\right)\right]}{\sqrt{1+\frac{\pi s^2}{4\alpha}}}, \tag{28}$$

which clearly tends to the Wigner surmise for the GOE as α approaches infinity. This formula provides a reasonable approximation for $P_{\mathrm{SOE}}(s,\alpha)$ at sufficiently large values of s for all values of $\alpha\neq 0$. In this respect, the asymptotic behavior of the superstatistical NNS distribution is given by

$$P_{\mathrm{SOE}}(s,\alpha) \sim C_1 \exp(-C_2 s^{2/3}), \tag{29}$$

where $C_{1,2}$ are constants, unlike that of the NNS distribution obtained by Tsallis' nonextensive statistics [14], which asymptotically decays according to a power law.

Figure 2 shows the evolution of $P_{\mathrm{SOE}}(s,\alpha)$ from a Wigner form towards a Poissonian shape as α decreases from ∞ to 0. This distribution behaves similarly but not quite exactly as any member of the large family of distributions. One of these is Brody's distribution [36], which is given by

$$P_{\mathrm{Brody}}(s,\gamma) = a_\gamma s^\gamma \exp[-a_\gamma s^{\gamma+1}/(\gamma+1)], \quad a_\gamma = \frac{1}{\gamma+1}\Gamma^{\gamma+1}\left(\frac{1}{\gamma+1}\right). \tag{30}$$

This distribution is very popular, but essentially lacks a theoretical foundation. It has been frequently used in the analysis of experiments and numerical experiments. The evolution of the Brody distribution during the stochastic transition is shown also in Fig. 2. The Brody distribution coincides with

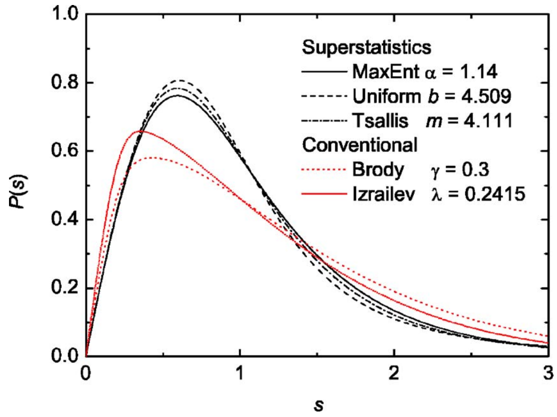

FIG. 3. (Color online) Comparison between the superstatistical and conventional NNS distributions having equal second moments.

the Wigner distribution if $\gamma=1$ and with Poisson's if $\gamma=0$. On the other hand, the superstatistical distribution at $\alpha=0$ is slightly different, especially near the origin. For example, one can use the small-argument expression of Mejer's G function to show that $\lim_{\alpha\to 0, s\to 0} P_{\mathrm{SOE}}(s,\alpha)=\pi/2$. In the midway of the stochastic transition, the agreement between the two distributions is only qualitative. At small s, the superstatistical distribution increases linearly with s while the increase of the Brody distribution is faster. The large s behavior is different as follows from Eqs. (29) and (30). The difference between the two distributions decreases as they approach the terminal point in the transition to chaos where they both coincide with the Wigner distribution.

The superstatistical NNS distribution for systems in the midway of a stochastic transition weakly depends on the choice of the parameter distribution. To show this, we consider other two spacing distributions, which have previously been obtained using other superstatistics [17,18,31]. The first is derived from the uniform distribution, considered in the paper of Beck and Cohen [19]. The second is obtained for a χ^2 distribution of the parameter η, which is known to produce Tsallis' nonextensive theory. In the latter case, we qualify the NNS distribution by the parameter $m=2/q-1-d-2$, where q is Tsallis' entropic index and d is the dimension of the Hamiltonian random matrix. This behavior is quite different from the conventional NNS which are frequently used in the analysis of experiments and nuclear experiments, namely Brody's and Izrailev's [37]. The latter distribution is given by

$$P_{\mathrm{Izrailev}}(s,\lambda)=As^{\lambda}\exp\left(-\frac{\pi^2\lambda}{16}s^2-\frac{\pi}{4}(B-\lambda)s\right), \quad (31)$$

where A and B are determined for the conditions of normalization and unit mean spacing. Figure 3 demonstrates the difference between the superstatistical and conventional distribution in midway between the ordered and chaotic limits. The figures compares these distributions with parameters that produce equal second moments. The second moment of the Brody distribution is given by

$$\langle s^2\rangle_{\mathrm{Brody}}=\frac{\Gamma\left(1+\frac{2}{\gamma+1}\right)}{\Gamma^2\left(1+\frac{1}{\gamma+1}\right)}. \quad (32)$$

We take $\gamma=0.3$, calculate $\langle s^2\rangle_{\mathrm{Brody}}$, and use the corresponding expressions for the second moment of the other distributions to find the value of their tuning parameters that makes them equal to $\langle s^2\rangle_{\mathrm{Brody}}$. The comparison in Fig. 3 clearly shows that, while the considered three superstatistical distributions are quite similiar, they considerably differ from Brody's and Izrailev's distributions.

The superstatistical distribution $P_{\mathrm{SOE}}(s,\alpha)$ can still be useful at least when Brody's distribution does not fit the data satisfactorily. As an example, we consider a numerical experiment by Gu *et al.* [38] on a random binary network. Impurity bonds are employed to replace the bonds in an otherwise homogeneous network. The authors of Ref. [38] numerically calculated more than 700 resonances for each sample. For each impurity concentration p, they considered 1000 samples with totally more than 700 000 levels computed. Their results for four values of concentration p are compared with both the Brody and superstatistical distribution in Fig. 4. The high statistical significance of the data allows us to assume the advantage of the superstatistical distribution for describing the results of this experiment.

B. Unitary ensembles

Now we calculate the superstatistical NNS distribution for a mixed system without time-reversal symmetry. Chaotic systems belonging to this class are modeled by GUE for which the Wigner surmise reads

$$P_{\mathrm{GUE}}(s,D)=\frac{32}{\pi^2D^3}s\exp\left(-\frac{4}{\pi D^2}s^2\right). \quad (33)$$

We again assume that the local-mean-spacing distribution $f(D)$ is given by Eq. (13). The superstatistics generalization of this distribution is obtained by substituting (33) into (22),

$$P_{\mathrm{SUE}}(s,\alpha)=\frac{32\alpha^3}{\pi^2D_0^3G_{03}^{30}\left(\alpha^3|0,\frac{1}{2},1\right)}s^2G_{03}^{30}\times\left(\alpha^3+\frac{4\alpha^2}{\pi D_0^2}s^2\middle|-1,-\frac{1}{2},0\right), \quad (34)$$

where D_0 is given by (14). At large values of z, we use the large z asymptotic formula for the G function to obtain

$$P_{\mathrm{SUE}}(s,\alpha)\approx\frac{32}{\pi^2}s^2\frac{\exp\left[-3\alpha\left(\sqrt[3]{1+\frac{4s^2}{\pi\alpha}}-1\right)\right]}{\left(1+\frac{4s^2}{\pi\alpha}\right)^{5/6}}, \quad (35)$$

which clearly tends to the Wigner surmise for the GUE as α approaches infinity as in the case of a GOE.

Figure 5 shows the behavior of $P_{\mathrm{SUE}}(s,\alpha)$ for different values of α ranging from 0 to ∞ (the GUE). As in the case of the orthogonal universality, the superstatistical distribution is

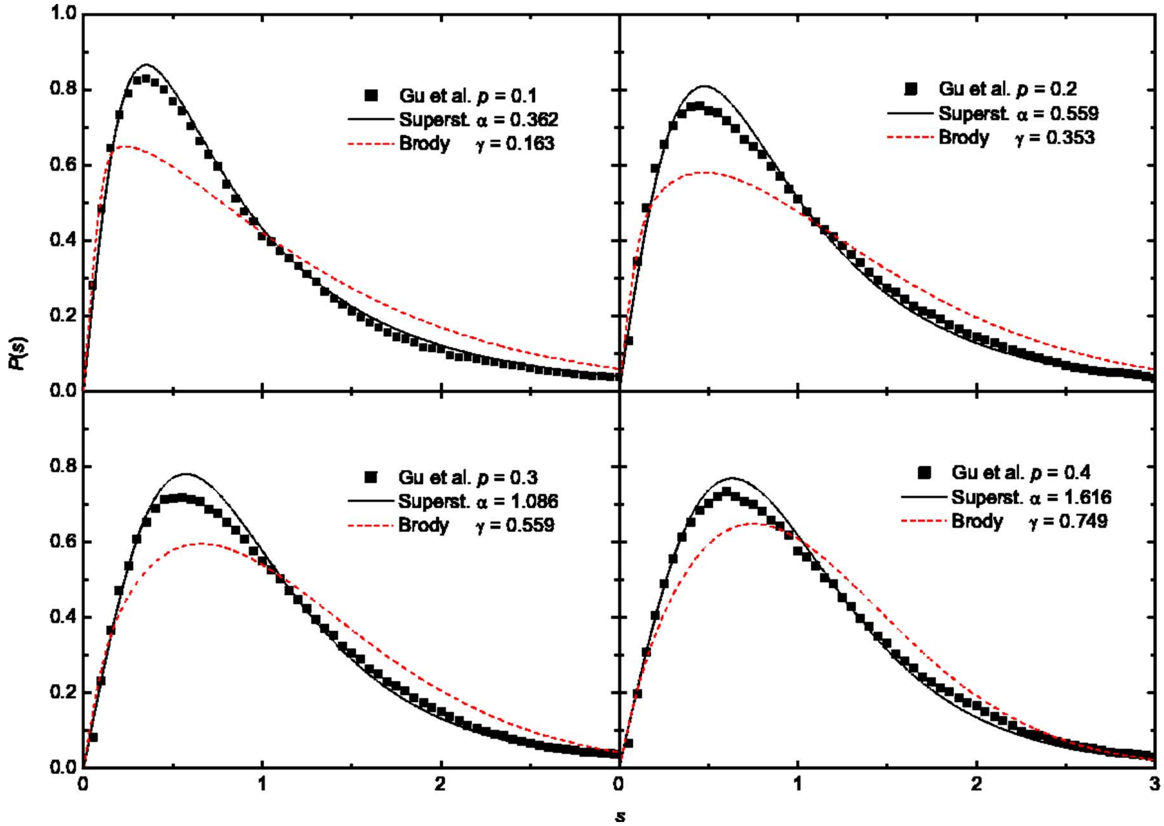

FIG. 4. (Color online) NNS distributions of geometrical resonances in random network, calculated by Gu *et al.* [38] compared with the Brody and superstatistical distributions.

not exactly Poissonian when $\alpha=0$. Using the small argument behavior of Mejer's G function, one obtains $\lim_{\alpha\to 0, s\to 0} P_{\text{SOE}}(s,\alpha)=4/\pi$.

V. TWO-LEVEL CORRELATION FUNCTION

The two-level correlation function is especially important for the statistical analysis of level spectra [4]. It is also directly related to other important statistical measures, such as the spectral rigidity Δ_3 and level-number variance Σ^2. These quantities characterize the long-range spectral correlations which have little influence on NNS distribution.

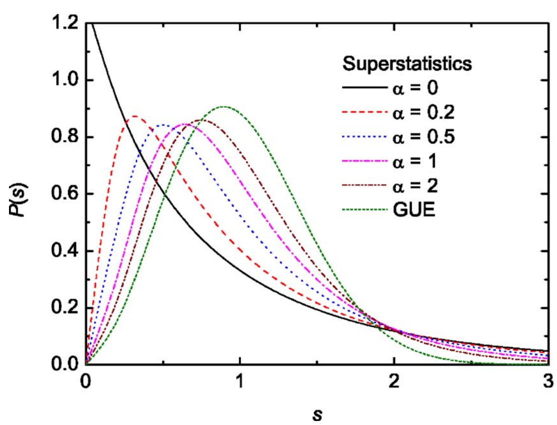

FIG. 5. (Color online) NNS distributions obtained by the superstatistics method for systems undergoing a transition from the GUE statistics to the Poissonian.

The two-level correlation function $R_2(E_1,E_2)$ is obtained from the eigenvalue joint distribution function $P_\beta^{(G)} \times(\eta,E_1,\ldots,E_N)$ by integrating over all eigenvalues except two. It is usually broken into connected and disconnected parts. The disconnected part is a product of two-level densities. On the unfolded spectra, the corresponding two-level correlation function can be written as [1,4].

$$X_2(\xi_1,\xi_2) = D^2 R_2(D\xi_1, D\xi_2). \tag{36}$$

Here the disconnected part is simply unity and the connected one, known as the two-level cluster function, depends on the energy difference $r=\xi_1-\xi_2$ because of the translation invariance. One thus writes

$$X_2(r) = 1 - Y_2(r). \tag{37}$$

The absence of all correlation in the spectra in the case of the Poisson regularity is formally expressed by setting all k-level cluster functions equal 0, and therefore

$$X_2^{\text{Poisson}}(r) = 1. \tag{38}$$

We shall here consider the unitary class of the symmetry. For a GUE, the two-level cluster function is given by

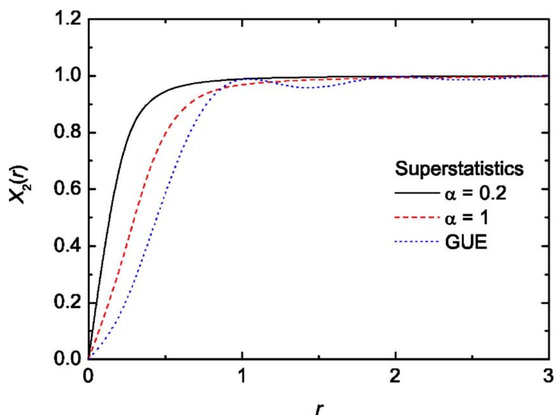

FIG. 6. (Color online) Two-level correlation functions obtained by the superstatistics method for systems undergoing a transition from the GUE statistics to the Poissonian.

$$Y_2^{\text{GUE}}(r) = \left(\frac{\sin \pi r}{\pi r}\right)^2. \tag{39}$$

The two-level correlation function for mixed systems described by the superstatistics formalism is given, using Eqs. (7) and (26), by

$$X_2^{\text{SUE}}(r) = \frac{1}{\langle D^{-2}\rangle}\int_0^\infty f(D)\frac{1}{D^2}X_2^{\text{GUE}}\left(\frac{r}{D}\right)dD, \tag{40}$$

where we divide by $\langle D^{-2}\rangle$ in order to get the correct asymptotic behavior of $X_2(r)\to 1$ as $r\to\infty$. Unfortunately, we were not able to evaluate this integral analytically in a closed form. The results of the numerical calculation of $X_2^{\text{SUE}}(r)$ for α=0.5, 1 and ∞ (the GUE) are given in Fig. 6.

VI. SUMMARY AND CONCLUSION

We have constructed a superstatistical model that allows us to describe systems with mixed regular-chaotic dynamics within the framework of RMT. The superstatistics arise out of a superposition of two statistics, namely one described by the matrix-element distribution $\exp[-\eta\,\text{Tr}(H^\dagger H)]$ and another one by the probability distribution of the characteristic parameter η. The latter defines the energy scale; it is proportional to the inverse square of the local mean spacing D of the eigenvalues. The proposed approach is different from the usual description of mixed systems, which model the dynamics by ensembles of deformed or banded random matrices. These approaches depend on the basis in which the matrix elements are evaluated. The superstatistical expressions depend on $\text{Tr}(H^\dagger H)$ which is invariant under base transformation. The model represents the spectrum of a mixed system as consisting of an ensemble of subspectra to which are associated different values of the mean level spacing D. The departure of chaos is thus expressed by introducing correlations between the matrix elements of RMT. Spectral characteristics of mixed systems is obtained by integrating the respective quantities corresponding to chaotic systems over all values of D. In this way, one is able to obtain entirely new expressions for the NNS distributions and the two-level correlation functions for mixed systems. These expressions reduce to those of RMT in the absence of the fluctuation of the parameter D, when the parameter distribution is reduced to a δ function. They can be used to reproduce experimental results for systems undergoing a transition from the statistics described by RMT towards the Poissonian level statistics, especially when conventional models fail. This has been illustrated by an analysis of a high-quality numerical experiments on the statistics of resonance spectra of disordered binary networks.

APPENDIX

For sake of completeness, we give in this appendix the definition of the Meijer G function as well as some of its properties, which have been used in the present paper. Meijer's G function is defined by

$$G_{p,q}^{m,n}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_j+s)\prod_{j=1}^n \Gamma(1-a_j-s)}{\prod_{j=m+1}^q \Gamma(1-b_j-s)\prod_{j=n+1}^p \Gamma(a_j+s)} z^{-s}ds, \tag{A1}$$

where $0\leqslant n\leqslant p$ and $0\leqslant m\leqslant q$ while an empty product is interpreted as unity. The contour L is a loop beginning and ending at $-\infty$ and encircling all the poles of $\Gamma(b_j+s), j=1,\ldots,m$ once in the positive direction but none of the poles of $\Gamma(1-a_j-s), j=1,\ldots,n$. Various types of contours, existence conditions, and properties of the G function are given in [39]. The way by which integrals of the type considered in this paper are expressed in terms of the G functions are described in [40].

The asymptotic behavior of Meijer's G function, as $|z|\to\infty$, is given by [41]

$$G_{p,q}^{m,n}\left(z\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right) \sim \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} z^{\theta}\exp(-\sigma z^{1/\sigma}), \tag{A2}$$

where $\sigma=q-p>0$, and $\sigma\theta=\frac{1}{2}(1-\sigma)+\Sigma_{j=1}^q b_j-\Sigma_{j=1}^p a_j$. In particular, the G function that appears in this paper

$$G_{0,3}^{3,0}(z|b_1,b_2,b_3) = \frac{1}{2\pi i}\int_L \frac{1}{\Gamma(1-b_1-s)\Gamma(1-b_2-s)\Gamma(1-b_3-s)} z^{-s}ds, \tag{A3}$$

has the following asymptotic behavior:

$$G_{0,3}^{3,0}(z|b_1,b_2,b_3) \sim \frac{2\pi}{\sqrt{3}} z^{(b_1+b_2+b_3-1)/3}\exp(-3z^{1/3}). \tag{A4}$$

On the other hand, the small z behavior of Meijer's G function [42] is given by

$$G_{p,q}^{m,n}\left(z\middle|\begin{matrix}a_1,\ldots a_n,a_{n+1},\ldots,a_p\\ b_1,\ldots b_m,b_{m+1},\ldots,b_q\end{matrix}\right) = \sum_{k=1}^{m}\frac{\prod_{\substack{j=1\\ j\neq k}}^{m}\Gamma(b_j-b_k)\prod_{j=1}^{n}\Gamma(1-a_j-b_k)}{\prod_{j=n+1}^{p}\Gamma(a_j-b_k)\prod_{j=m+1}^{q}\Gamma(1-b_j-b_k)}z^{b_k} \times\left[1+\frac{\prod_{j=1}^{p}(1-a_j-b_k)}{\prod_{j=1}^{n}(1-b_j-b_k)}(-1)^{-m-n+p}z+\cdots\right]. \tag{A5}$$

Thus, the leading term in the expansion of $G_{0,3}^{3,0}(z|b_1,b_2,b_3)$ in powers of z is given by

$$G_{0,3}^{3,0}(z|b_1,b_2,b_3)\approx\Gamma(b_2-b_1)\Gamma(b_3-b_1)z^{b_1}, \tag{A6}$$

where b_1 is the smallest of b_i.

The implementation of Meijer's G function in Mathematica [42] constitutes an additional utility for analytic manipulations and numerical computations involving this special function.

[1] M. L. Mehta, *Random Matrices*, 2nd ed (Academic, New York, 1991).
[2] F. Haake, *Quantum Signitures of Chaos* (Springer, Heidelberg, 1991).
[3] E. V. Berry and M. Tabor, Proc. R. Soc. London, Ser. A **356**, 375 (1977).
[4] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[5] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960).
[6] M. S. Hussein and M. P. Pato, Phys. Rev. Lett. **70**, 1089 (1993); Phys. Rev. C **47**, 2401 (1993).
[7] M. S. Hussein and M. P. Pato, Phys. Rev. Lett. **80**, 1003 (1998).
[8] Arvid Lundervold, AIM Conference Proceedings **777**, 171 (1996).
[9] G. Casati, L. Molinari, and F. Izrailev, Phys. Rev. Lett. **64**, 1851 (1990).
[10] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67**, 2405 (1991).
[11] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[12] C. Tsallis, Lect. Notes Phys. **560**, 3 (2001).
[13] J. Evans and F. Michael, e prints cond-mat/0207472 and cond-mat/0208151.
[14] F. Toscano, R. O. Vallejos, and C. Tsallis, Phys. Rev. E **69**, 066131 (2004).
[15] A. C. Bertuola, O. Bohigas, and M. P. Pato, Phys. Rev. E **70**, 065102(R) (2004).
[16] F. D. Nobre and A. M. C. Souza, Physica A **339**, 354 (2004).
[17] A. Y. Abul-Magd, Phys. Lett. A **333**, 16 (2004).
[18] A. Y. Abul-Magd, Phys. Rev. E **71**, 066207 (2005).
[19] C. Beck and E. G. D. Cohen, Physica A **322**, 267 (2003).
[20] S. Abe, Phys. Rev. E **66**, 046134 (2002).
[21] A. M. C. Souza and C. Tsallis, Phys. Lett. A **319**, 273 (2003).
[22] C. Tsallis and A. M. C. Souza, Phys. Rev. E **67**, 026106 (2003); Phys. Lett. A **319**, 273 (2003).
[23] E. G. D. Cohen, Physica D **193**, 35 (2004).
[24] C. Beck, Physica D **193**, 195 (2004).
[25] C. Beck, Europhys. Lett. **64**, 151 (2003).
[26] F. Sattin and L. Salasnich, Phys. Rev. E **65**, 035106(R) (2003).
[27] F. Sattin, Phys. Rev. E **68**, 032102 (2003).
[28] A. M. Reynolds, Phys. Rev. Lett. **91**, 084503 (2003).
[29] M. Ausloos and K. Ivanova, Phys. Rev. E **68**, 046122 (2003).
[30] C. Beck, Physica A **331**, 173 (2004).
[31] A. Y. Abul-Magd, Physica A **361**, 41 (2006).
[32] R. Balian, Nuovo Cimento **57**, 183 (1958).
[33] A. Abd El-Hady, A. Y. Abul-Magd, and M. H. Simbel, J. Phys. A **35**, 2361 (2002); A. Y. Abul-Magd, C. Dembowski, H. L. Harney, and M. H. Simbel, Phys. Rev. E **65**, 056221 (2002); A. Y. Abul-Magd and M. H. Simbel, *ibid.* **70**, 046218 (2004).
[34] A. C. Bertuola, J. X. de Carvalho, M. S. Hussein, M. P. Pato, and A. J. Sargeant, Phys. Rev. E **71**, 036117 (2005).
[35] C. E. Porter, *Statistical Theory of Spectra; Fluctuation* (Academic Press, New York, 1965).
[36] T. Brody, Lett. Nuovo Cimento Soc. Ital. Fis. **7**, 482 (1973).
[37] F. M. Izrailev, Phys. Lett. A **134**, 13 (1988).
[38] Y. Gu, K. W. Yu, and Z. R. Yang, Phys. Rev. E **65**, 046129 (2002).
[39] A. M. Mathai, *A Handbook of Special Functions* (Clarendon, Oxford, 1993).
[40] A. M. Mathai and H. J. Haubold, Astrophys. Space Sci. **282**, 265 (2002).
[41] Y. M. Luke, *The Special Functions and Their Applications* (Academic Press, New York, 1969), Vol. 1.
[42] Wolfram Research's Mathematical Functions 2002, http://functions.wolfram.com